Ordinary Differential Equations

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The exam consists of 5 exercises. You have 180 minutes to do these exercises. The maximum number of points for each part of the exercises is given at the beginning of the exercise. The total number of points you can get is 100. You get 10 points for free.

1. **Exact differential equations [16 Points.]**

   Check whether the following differential equations are exact, determine an integrating factor (in case they are not exact) and find the general solution.

   (a) $y'y + x = 0$,

   (b) $y' \sin x - 2y \sin x + y \cos x = 0$.

   [Hint: the second differential equation admits an integrating factor which is a function of x only.]

2. **Linear second order differential equations [8 Points.]**

   Solve the second order differential equation

   $$2y'' - y' - y - e^x \cos(2x) = 0.$$

3. **Existence and uniqueness of solutions [15+10 Points.]**

   (a) Let a and b be positive constants. Consider the initial value problem $y' = f(x, y)$, $y(0) = 0$ for the following functions $f : [0, a] \times [-b, b] \to \mathbb{R}$, $(x, y) \mapsto f(x, y)$:

      i. $f(x, y) = x \sin y + y \cos x$,

      ii. $f(x, y) = x^2 e^{x+y}$,

      iii. $f(x, y) = p(x)y + q(x)$ where p and q are continuous functions on the interval $[0, a]$.

      For each case, compute a Lipschitz constant (with respect to y) and determine an interval $[0, \alpha]$ in which you can guarantee the existence of a unique solution without solving the initial value problem.

   (b) State Peano's Existence Theorem and give an example of an initial value problem of the form $y' = f(x, y)$, $y(x_0) = y_0$ which has a solution that is not unique. Write down two different solutions for the initial value problem in your example.

4. **Linear systems [12+14 Points.]**

(a) Give the general solution of the linear system $\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ for the following matrices A:

i.
$$A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}.$$

ii.
$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

(b) Consider the system

$$\begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = A \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \text{with } A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

i. Find the general solution by computing the eigenvalues and eigenvectors of A. Use the general solution to form a fundamental matrix Ψ.

ii. Compute the principal fundamental matrix $\Phi(x) = \exp(Ax)$ from exponentiating Ax. [Hint: you may use that $\cosh x = (\mathrm{e}^x + \mathrm{e}^{-x})/2$ and $\sinh x = (\mathrm{e}^x - \mathrm{e}^{-x})/2$.]

iii. Find a non-singular 3×3 matrix C such that $\Psi = \Phi C$.

5. **Periodic linear systems [3+5+7 Points.]**

Consider the differential equation $y'' + y = \cos 2x$.

(a) Show that $y(x) = -\frac{1}{3}\cos(2x)$ is a solution of the differential equation.

(b) Determine the general solution of the corresponding homogeneous differential equation $y'' + y = 0$.

(c) In the lecture the following statement (Corollary 20.4) was proved: *An ω-periodic linear system*[1] *has a unique ω-periodic solution if and only if for the corresponding homogeneous system, the trivial solution is the only ω-periodic solution.* Use this statement to comment on the uniqueness of periodic solutions of period π and period 2π of the given differential equation.

[1] We here assume that the coefficient functions are not only periodic but also continuous to guarantee existence and uniqueness of solutions.